Permutation

Let S be a nonempty finite set. A bijective map $f: S \to S$ is said to be a Permutation on S.

Let, $S = \{a_1, a_2, \dots, a_n\}$. Then the number of bijections from S onto S is $n!$. Let one such bijection be f that maps a_i to $f(a_i)$. This permutation f is denoted by

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(a_1) & f(a_2) & \cdots & f(a_n) \end{pmatrix}.$$

The id. mapping is is also a bijective map. is is said to be the identity permutation and it is denoted by i.

$$i = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}.$$

Let, $S = \{1, 2, 3, 4\}$ and a permutation f be defined by $f(1) = 2$, $f(2) = 4$, $f(3) = 1$, $f(4) = 3$

Then f is $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 3 & 4 & 2 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

or etc.

Multiplication of permutations. Let, $f, g : S \to S$ be two permutations on S. Let, $f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(a_1) & f(a_2) & \cdots & f(a_n) \end{pmatrix}$ $g = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ g(a_1) & g(a_2) & \cdots & g(a_n) \end{pmatrix}$

$$fg = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f[g(a_1)] & f[g(a_2)] & \cdots & f[g(a_n)] \end{pmatrix} \quad gf = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ g[f(a_1)] & g[f(a_2)] & \cdots & g[f(a_n)] \end{pmatrix}$$

Now, $fg \neq gf$ in general. (Since, comp. of map is not commutative)

But, $f(gh) = (fg)h$, for any three permutation f, g, h on S.

Inverse of a Permutation. Let f be a permutation on S. Since, f is a bijective map, it admits a unique inverse $f^{-1} : S \to S$. and is also bij. So, f^{-1} is a permutation on S.

So, if $f = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ f(a_1) & f(a_2) & \cdots & f(a_n) \end{pmatrix}$ $f^{-1} = \begin{pmatrix} f(a_1) & f(a_2) & \cdots & f(a_n) \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$

① Let $S = \{1, 2, 3, 4\}$ and $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$.

$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

Find, fg, gf, $f(gh)$, $(fg)h$, and f^{-1}

Cycle :

Let, $S = \{a_1, a_2, \dots, a_n\}$. A permutation $f: S \to S$ is said to be a cycle of length 'n', or an n-cycle if there are 'n' elements $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ in S s.t. $f(a_{i_1}) = a_{i_2}, f(a_{i_2}) = a_{i_3}, \dots$ $f(a_{i_{n-1}}) = f(a_{i_n})$, $f(a_{i_n}) = a_{i_1}$ and $f(a_j) = a_j$, $j \neq i_1, i_2, \dots, i_n$

This cycle is denoted by $(a_{i_1}, a_{i_2}, \dots, a_{i_n})$ or in any other form provided the elements appear in a fixed cyclic order. $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ are called the elements of the cycle. Two cycles f and g on the same set S are said to be disjoint if they have no common elements.

EX: $S = \{1, 2, 3, 4\}$ $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$

$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

f can be written as, $(2\ 3\ 4)$ (3-cycle)

g " " " " $(2\ 4)$ (2-cycle)

Th: Every permutation on a finite set is either a cycle or it can be expressed as a product of disjoint cycles.

Order of a Permutation :

Let 'f' be a Permutation on a finite set S. The order of 'f' is the least positive integer 'n' s.t. $f^n = i$, i being the identity Permutation.

Th: The order of an n-cycle is 'n'.

Th: The order of a permutation on a finite set is the l.c.m. of the lengths of its disjoint cycles.

Th: A 2-cycle is called a transposition.

Th: Every permutation on a finite set can be expressed as a product of transpositions.

Def^n → A permutation is said to be even if it can be expressed as the product of an even number of transpositions and odd if it can be expressed as the product of an odd number of transpositions.

Th: The number of even permutations on a finite set (containing at least two elements) is equal to the number of odd permutations on it.

① Find fg, gf, f^{-1}, g^{-1} where

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix} \ \& \ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 5 & 3 & 2 \end{pmatrix}$$

$$f^{-1} = \begin{pmatrix} 2 & 4 & 3 & 5 & 6 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 2 & 4 & 5 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} 1 & 6 & 4 & 5 & 3 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 3 & 4 & 2 \end{pmatrix}$$

$$fg = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 5 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 6 & 3 & 4 \end{pmatrix}$$

$$gf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & [illegible] & [illegible] & [illegible] & 2 & 1 \end{pmatrix}$$

(2) Find the order of the permutations.

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix}$

$= (1\ 4\ 5)\ (2\ 6)\ (3)$

$\cancel{= (1\ 5)\ (1\ 4)\ (2\ 6)}$

Order $= 3 \cdot 2 \cdot 1 = 6.$

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 1 & 3 & 2 & 8 & 6 & 7 \end{pmatrix}$ (HW). (6).

Equipotent sets.

Let A and B be two nonempty sets. 'A' is said to be equipotent with B if there exists a bijective mapping 'f' from A to B. We write it as $A \sim B$. If 'A' is equipotent with 'B', then 'B' is equipotent with 'A'.

* Two equipotent sets are said to have the same cardinal number. The cardinal number of a set A is denoted by card(A).

So, A and B are equipotent sets iff card(A) = card(B)

card $(\phi) = 0$.

The cardinality of an infinite set is said to be a transfinite cardinal number.

card $(\mathbb{N}) = d$.

card $(\mathbb{R}) = c$.