Elasticity

→ A rigid body is defined as one in which the distances between any two of its points remain unaffected → Ideal Definition. even when forces act upon it.

→ But all ~~bodi~~ bodies can be deformed by suitable forces. Deformation may be a change in size, shape or both.

→ **What is Elasticity ?**

Elasticity is a property by virtue of which a body undergoes a change in shape, size or both due to external forces, tends to regain its original shape, size as soon as the force is removed.

→ Perfectly Elastic → Body regain its original condition completely.

→ Perfectly Plastic → Fully retain the deformation even when the ~~deformation~~ deforming forces are withdrawn,

→ No body is perfectly elastic when subjected to very great forces.

→ every substance is elastic to some degree however small,

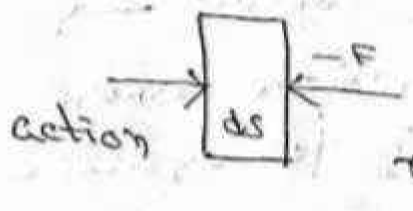
→ Homogeneous Body: A body is said to be homogeneous if every part of it is composed of the same substance.

→ Isotropic Body → if it possessing identical properties in every direction. if not it is called anisotropic (Most crystals are anisotropic)

We shall consider here only homogeneous and isotropic materials

Stress and Strain.

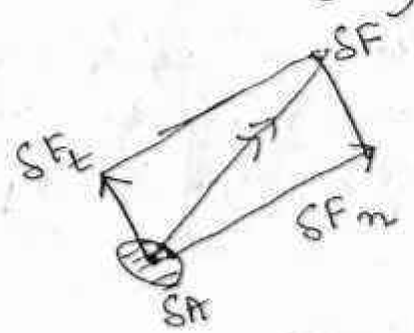
Whenever a force system in equilibrium acts upon a body, it is said to be under a state of strain, a reaction force formed inside the object. one part imparted force on another part → Surface force.



equilibrium.

imaginary surface → force from both sides, molecules from one side gives force on another side of that imaginary surface. range $10^{-7} - 10^{-8}$ cm.

(molecules are closed within very short range).



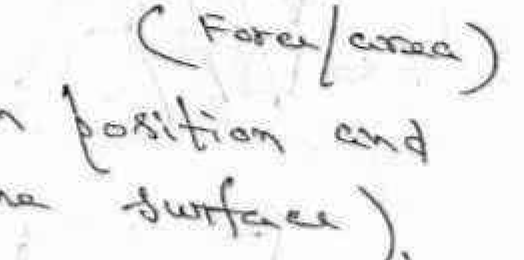
$$\text{Stress} = \lim_{\delta A \to 0} \frac{\delta F}{\delta A} = \frac{dF}{dA}$$

(Force / area)

Stress depends on position and orientation of the surface).

Three kinds of stresses

i) Tensile stress

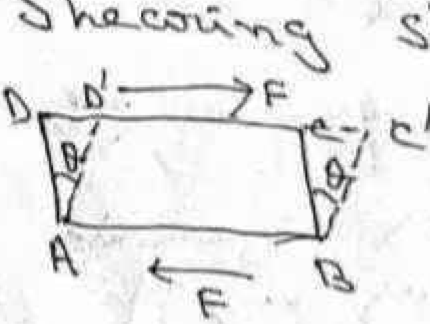


(bar of uniform cross section be subjected to equal and opposite pulls)

ii) compressive stress →

(bar subjected to pushes, instead of pulls at the ends)

iii) Shearing stress →

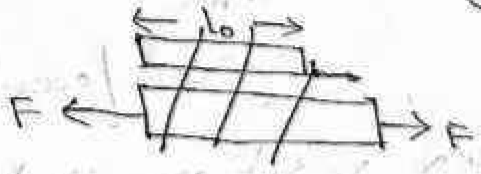


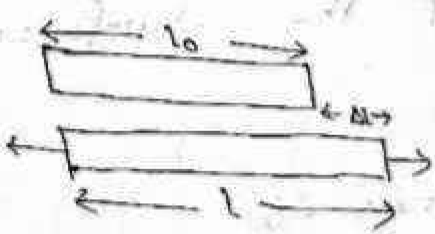
Strain-

A body is under strain when a system of forces in equilibrium acts on it.
The strained condition is manifested by the deformation the body undergoes.

Diff. kinds of strain

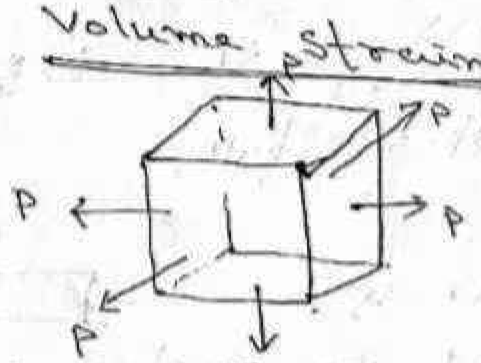
i) Tensile / longitudinal strain.
(corresponds to Tensile stress).





Tensile strain
$= \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0}$
(only for solids).

ii) Volume strain



uniform pressure P acting outwards perpendicular to each surface of the cube.

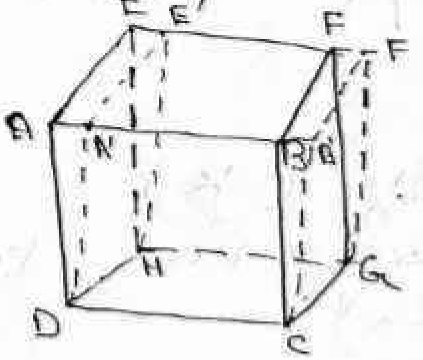
initial volume v_0
Final " $v_0 + \Delta v$

volume strain $= \frac{\Delta v}{v_0}$

(if v acts inwards, volume decreasing $-\Delta v$)

(5) volume stress char. of solid, liquid, gaseous state

iii) Shearing strain (corresponds to shearing stress).



Shearing strain

$\theta \simeq \tan\theta$

$= \frac{BB'}{BC} = \frac{AA'}{AD}$

(only characteristics of solids).

Hooke's law: (Elastic limit).

For any elastic deformation, the stress is proportional to the corresponding strain, within a certain limit of stress, which is called elastic limit.

$\therefore$ stress $\propto$ strain

$\frac{\text{stress}}{\text{strain}}$ = constant.

(This is Hooke's Law).

This constant is called elastic constant.

a) Young's Modulus (corresponds to longitudinal stress and strain).

$$Y.M\ (Y) = \frac{F/A}{\Delta l/l} = \frac{Fl}{A\Delta l}$$

b) Bulk modulus (K)
(corresponds to volume stress and strain).

$$K = \frac{F/A}{\Delta V/V} = \frac{P}{\Delta V/V}$$

~~K~~

$$C = \frac{1}{K}$$ (compressibility of the material).

c) Rigidity modulus or Shearing modulus G.

$$G = \frac{\text{Shearing stress}}{\text{" strain}}$$

$$= \frac{F/A}{\theta} = \frac{F/A}{\tan\theta}$$

d) Axial modulus (χ)

$$\chi = \left.\frac{\text{longitudinal stress}}{\text{longitudinal strain}}\right|_{\text{lateral Strain}=0}$$

(x ~~is im~~ and E are similar but
$E \rightarrow$ lateral strain are allowed
$x \rightarrow$ " " " not ")

it is important constant in the propagation of longitudinal waves through an infinitely extended medium, e.g. longitudinal earthquake waves through the earth surface get alternately compressed and dilated in the wave - propagation direction, but prevents lateral compression.

Poisson's ratio

(if a wire gets stretched, it becomes thinner, when compressed, it gets thicker)

Within elastic limit,

$$\mu = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \text{Poisson's ratio}$$

(depends on nature of the material, but not an elastic constant.)

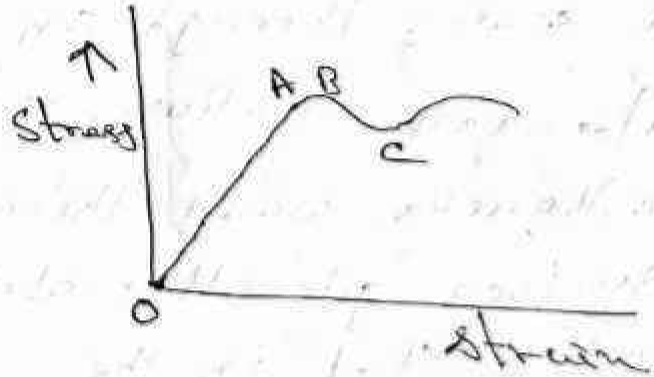
⑧

Elastic behaviour of a stretched string

According to Hooke's law

Stress = cont. Strain

(within elastic limit)



if we draw (stress − strain) graph.

OA → elastic limit (straigh line)

$y = mx$ ↓ elastic cont.)

If stress is increased beyond A, to B, the wire cannot reach the original state even after withdrawal of stress.

and finally at C, the specimen (say, wire) breaks)

Dimension

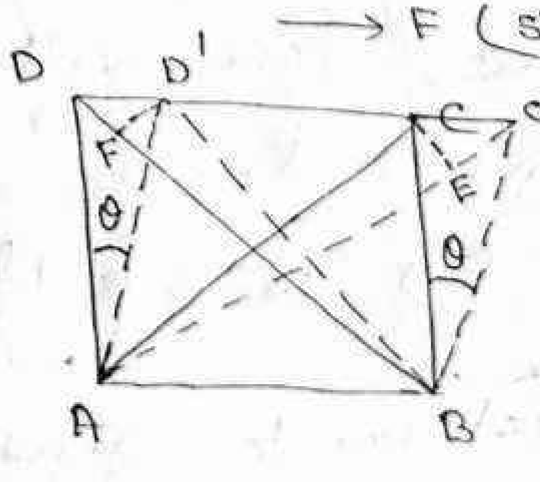
Stress = $\frac{\text{Force}}{\text{Area}}$

unit = dyne/cm^2 or N/m^2 $= ML^{-1}T^{-2}$

Strain = dimension less quantity

⇒ no unit

A shear is equivalent to ~~an~~ equal elongation and contraction at right angles to each other.



(AC extended to AC')
BD compressed to BD')
$\theta \approx$ very small.
$CE \perp AC'$
$\therefore AC \simeq AE$

Similarly $BF \simeq BD'$

Strain produced along

$AC \Rightarrow \frac{EC'}{AC} \rightarrow (1)$

along $BD \rightarrow \frac{DF}{BD} \rightarrow (2)$.

Now,

$\frac{EC'}{CC'} = \cancel{\tan}\ \cos 45 = \frac{1}{\sqrt{2}}$ ($\theta \to$ v. small $\therefore \angle EC'C = 45°$)

$EC' = \frac{CC'}{\sqrt{2}}$, $\frac{CC'}{CB} = \tan\theta$

$= \frac{CB \cdot \tan\theta}{\sqrt{2}}$, $\frac{CB}{AC} = \cancel{\tan}\ \sin 45 = \frac{1}{\sqrt{2}}$

$= \frac{AC}{\sqrt{2}} \cdot \frac{\tan\theta}{\sqrt{2}}$

$EC' = \frac{AC}{2} \tan\theta \approx AC \cdot \frac{\theta}{2}$

$\therefore \frac{EC'}{AC} = \frac{\theta}{2}$

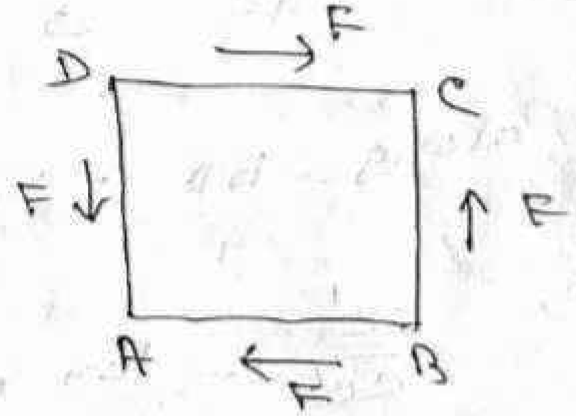
$\therefore$ From (1) strain $= \frac{\theta}{2}$

this is elongated strain along AC.

From similar calc. we will get compressive strain along BD will be $\frac{\theta}{2}$

Hence the above statement proves.

Physical explanation



i) Shearing force $\vec{F}$ applied along $\vec{DC}$ keeping AB fixed.

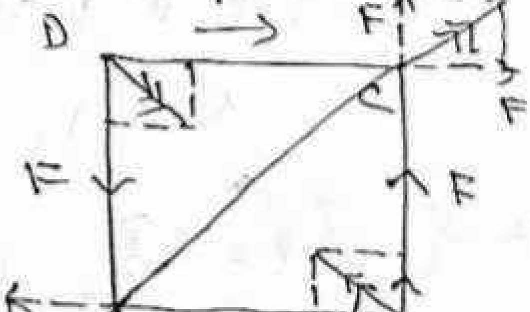
ii) internal reactive force will be along $\vec{BA}$ to keep it in equilibrium.

iii) So, two equal and opposite directional parallel force form a torque which tends to rotate the body in clockwise direction.

Now to ~~kee~~ prevent this rotation an internal torque ~~an~~ will be formed in anticlockwise direction. and this corresponds to parallel and opposite directional force F along $\overrightarrow{BC}$ and $\overrightarrow{DA}$. (See Fig).

<u>Now</u>

~~consider~~
concentrate on C point.



at C → two force $\vec{F}$ results extension of C point along outwards.

So, AC → will be extended.

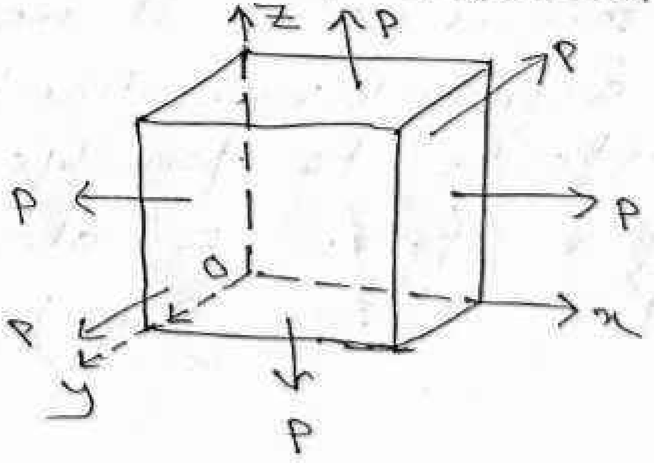
Simillarly BD will be compressed

and also, AC ⊥ BD

Proved

Relation between Elastic constants.

1) Relation between Y, K, μ



unit cube.

x, y, z axes are shown.

P force acts outw. perp. to every surface of the cube.

So, $(\vec{P}, \vec{P})$ along x, y, z axis

we know

$$Y = \frac{P}{\frac{\Delta l}{l}} = \frac{P}{\Delta l} \quad (\because l = 1 \text{ unit cube})$$

$$\Delta l = \frac{P}{Y}$$

Poisson's ratio

$$\mu = \frac{\text{Lateral strain}}{\text{longitudinal str.}}$$

Lateral strain = μ. longit strain

Now consider $\vec{P}, \vec{P}$ along two opposite direction of x - axis.

$$\therefore \Delta l = \frac{P}{Y} \quad (\text{along } x\text{-axis})$$

Lateral strain $\mu \Delta l = \mu \frac{P}{Y}$ (along y and z - axis.)

Similar strain will be produced for $(\vec{P}, \vec{P})$ along Y - axis

and $(\vec{P}, \vec{P})$ along Z - axis.

$\therefore$ Total elongation of length along x-axis $= \frac{P}{Y} - \mu\frac{P}{Y} - \mu\frac{P}{Y}$

$$= (1-2\mu)\frac{P}{Y}$$

From symmetry same elongation along Y and Z-axis also.

increase in length in each side

$$l' = 1 + (1-2\mu)\frac{P}{Y}$$

Since it is a cube, so

volume $= l'^3$

$$= \left[1 + (1-2\mu)\frac{P}{Y}\right]^3$$

$$= 1 + 3(1-2\mu)\frac{P}{Y} + 3(1-2\mu)^2\frac{P^2}{Y^2} + \text{cube term}$$

$$\simeq 1 + 3(1-2\mu)\frac{P}{Y} \quad \text{(neglect higher order)}$$

Change in volume $\Delta v = 3(1-2\mu)\frac{P}{Y}$

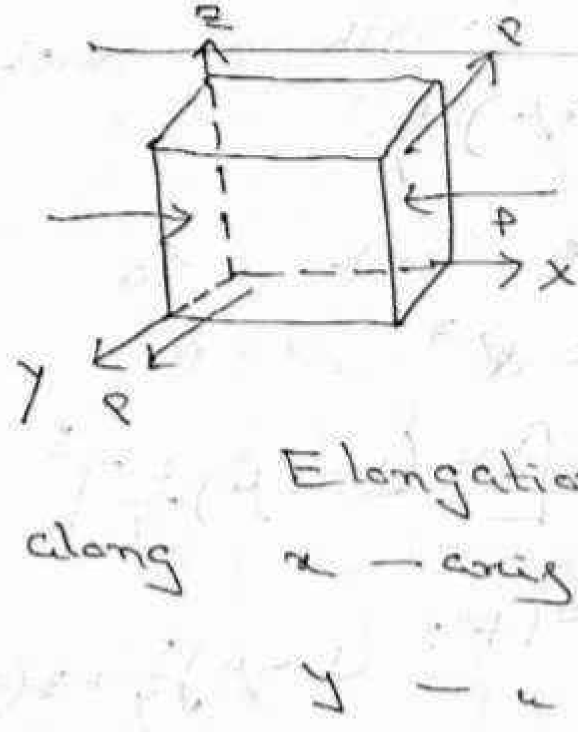
∴ Bulk modulus

$$K = \frac{P}{3(1-2\mu)\frac{P}{Y}}$$

$$K = \frac{Y}{3(1-2\mu)}$$

$$\boxed{Y = 3K(1-2\mu)} \rightarrow (A)$$

b) Relation betⁿ Y, G, μ



- compressive stress along x-axis
- Tensile stress along y-axis

Elongation along x-axis $= -\frac{P}{Y} - \mu\frac{P}{Y} = -\frac{P}{Y}(1+\mu)$

y – " $= \frac{P}{Y} + \mu\frac{P}{Y} = \frac{P}{Y}(1+\mu)$

equal compressive strain along x-axis and elongation along Y-axis produces a shear which is $\theta = \frac{2P}{Y}(1+\mu)$

rigidity modulus $= \frac{P}{\theta}$

$$= \frac{P}{\frac{2P}{Y}(1+\mu)}$$

$$G = \frac{Y}{2(1+\mu)}$$

$$\boxed{Y = 2G(1+\mu)} \longrightarrow (B)$$

c) <u>Relation betn G, K, μ</u>

From (A) $Y = 3K(1-2\mu)$

" (B) $Y = 2G(1+\mu)$

$$\therefore 3K(1-2\mu) = 2G(1+\mu)$$

$$\mu(2G+6K) = 3K-2G.$$

$$\mu = \frac{3K-2G.}{2G+6K}$$

$$= \frac{3K-2G.}{2(G+3K)}$$

$$\longrightarrow (C)$$

⇒ if $K >> G$

$$\mu = \frac{3K}{6K} = \frac{1}{2}$$

if $G >> K$

$$\mu = \frac{-2G}{2G} = -1$$

So, value of μ lies betn $\frac{1}{2}$ and -1

Most of the material have a μ - value betn .2 and .4

d) Relation betn Y, K, G.

From (A) $Y = 3K(1-2\mu)$

$$\therefore \mu = \frac{-Y+3K}{6K}$$

From (B) $Y = 2G(1+\mu)$

$$\mu = \frac{Y-2G}{2G}$$

$$\therefore \frac{-Y+3K}{6K} = \frac{Y-2G}{2G}$$

$$-\frac{Y}{6K} + \frac{1}{2} = \frac{Y}{2G} - 1$$

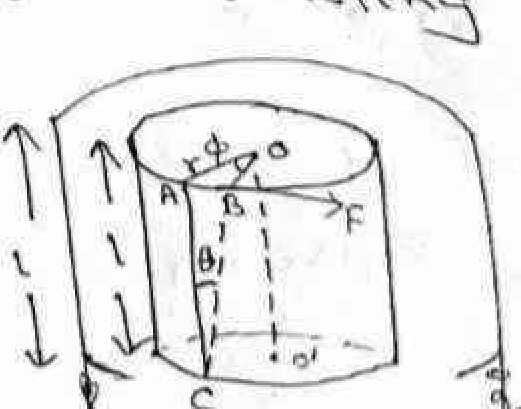
$$-\frac{1}{3K} + \frac{1}{Y} = \frac{1}{G} - \frac{2}{Y}$$

$$\frac{3}{Y} = \frac{1}{G} + \frac{1}{3K}$$

$$\boxed{\frac{9}{Y} = \frac{1}{K} + \frac{3}{G}}$$

Torsion of a cylinder / wire

If a cylinder, fixed at one end (say, lower end), be twisted about its axis at the other end by an external couple, applied in a plane perpendicular to the axis, there is no change in its length / radius; only particles of each cross-section are displaced relative to those in the adjacent cross-section — an example of pure shear. Due to elastic property of the material of the wire, a restoring : shearing couple is set up in the cylinder that balance external applied twisting couple.



a cylinder of length 'l', radius a. Lower end is fixed.

on upper surface a torque is applied, w.r.t the axis of the cylinder.

When the cylinder is twisted, each cross-section of the cylinder rotates in its own plane. This is angle of twist ϕ

consider an imaginary coaxial cylinder of length l, radius r $(r < a)$. AC is parallel to OO'. gets inclined to BC.

$$\angle ACB = \theta.$$

arc $AB = r\phi = l\theta$

$$\theta = \frac{r\phi}{l}$$

Take an small area α ~~on the~~ at A.

dx = length

dr = breadth

$$\alpha = dx\, dr$$

Tangential force F

Tangential Stress $= \frac{F}{\alpha}$

$$G = \frac{F/\alpha}{\theta} = \frac{F/(dr\, dx)}{\frac{r\phi}{l}}$$

$$\therefore F = \frac{G\phi}{l} r\, dr\, dx$$

Moment of the force

$$= F.r. = \frac{G\phi}{l} r^2\, dr\, dx$$

restoring torque over the whole surface

$$dN = \frac{G\phi}{l} r^2 dr \sum dx$$

$$= \frac{G\phi}{l} r^2 dr \, 2\pi r$$

$$= \frac{2\pi G\phi}{l} r^3 dr$$

Total torque over the whole cylinder

$$N = \int_0^a dN = \frac{2\pi G\phi}{4l} a^4$$

$$N = \frac{\pi G a^4}{2l} \phi$$

$C = \frac{N}{\phi} \rightarrow$ torsional rigidity

For hollow cylinder

$$N = \frac{\pi G\phi}{2l} \left(r_2^4 - r_1^4 \right).$$

$$C = \frac{\pi G}{2l} \left(r_2^4 - r_1^4 \right).$$

Torsional modulus ($C \rightarrow$ when $\phi = 1$, $l = 1$)

$$\Downarrow$$

$$\boxed{\frac{1}{2}\pi G a^4}$$

1. Diff. defn.
2. Hooke's Law, Poisson's ratio.
3. Shear is equivalent to equal elongatn and compression.
4. relatn betn Y, K, G, σ
5. Torsional rigidity

a) A hollow cylinder is stronger than a solid cylinder of same length, mass and material.

b) If a number of rods of torsional rigidity $c_1, c_2, \ldots$ are joined end to end, the torsional rigidity of the combination is given by the same formula as that of a number of electrical resistances in parallel.

$$N = c_1\phi_1 = c_2\phi_2 = \cdots = c_n\phi_n$$

$$\phi = \phi_1 + \phi_2 + \cdots + \phi_n$$

$$N\frac{1}{C} = N\left(\frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n}\right)$$

$$\frac{1}{C} = \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n}$$