Minkowski Space

- In classical physics time-co-ordinate is unaffected by a transformation from one inertial system to another. $(t' = t)$
- $t' \longrightarrow$ not dependent on space co-ord (x, y, z).
- In relativity
$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\beta^2}} , \quad \beta = \frac{v}{c}$$
$v \rightarrow$ velocity of prime frame w.r.t another inertial frame.

- Minkowski first treated space and time together in ~~diagram~~ geometric representation which is called space-time diagram.

- The co-ord of an event in relativity is represented by (x, y, z, t), but for simplicity we take only x-axis as space axis.
$$\therefore (x, t)$$

- Take space – axis horizontal
  time – " vertical.

- To keep dimension of the co-ord same, can take ct instead of t. $c \rightarrow$ vel. of light

- So, L.T $\Rightarrow$ $\beta = \frac{v}{c}$

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} = \frac{x - \frac{v}{c}\cdot ct}{\sqrt{1-\beta^2}} = \frac{x - \beta\omega}{\sqrt{1-\beta^2}}$$

||ly $$\boxed{x = \frac{x' + \beta\omega'}{\sqrt{1-\beta^2}}} \qquad \boxed{x' = \frac{x - \beta\omega}{\sqrt{1-\beta^2}}}$$

(1)

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\beta^2}} = \frac{ct - \frac{v}{c}x}{c\sqrt{1-\beta^2}}$$

$$ct' = \frac{ct - \beta x}{\sqrt{1-\beta^2}}$$

$$\boxed{\omega' = \frac{\omega - \beta x}{\sqrt{1-\beta^2}}}$$

||ly $$\boxed{\omega = \frac{\omega' + \beta x'}{\sqrt{1-\beta^2}}}$$

geometric Representation

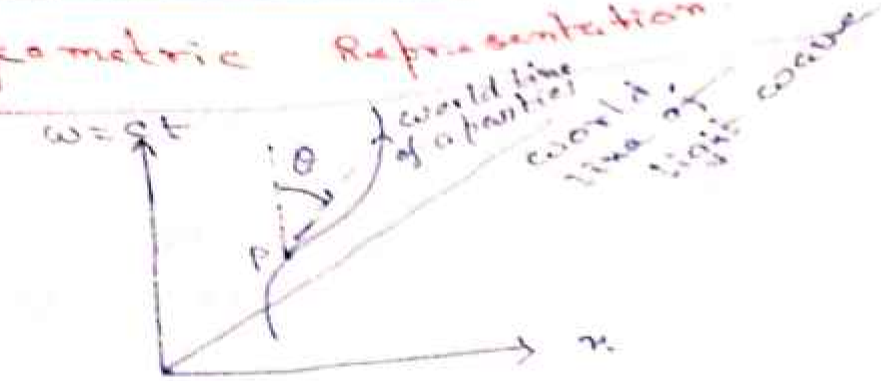


Fig 1

$(x-\omega) \rightarrow S$ frame $(x \perp \omega)$.

- Minkowski referred to space-time as the 'The World'.
- So, each event is a world point and collection of events is the motion of a particle in a world line
- The tangent at any pt P in world-line is inclined to time axis at an angle θ. $\theta < 45^\circ$.

because
$$\tan\theta = \frac{dx}{d\omega} = \frac{dx}{dt}\cdot\frac{dt}{d\omega} = \frac{1}{c}\frac{dx}{dt}$$

$$\left[\because \omega = ct, \ \frac{d\omega}{dt} = c\right]$$

$\frac{dx}{dt} = u \rightarrow$ vel. of a material particle.

$u < c$

$\therefore \tan\theta < 1, \quad \theta < 45^\circ$.

- For light wave
$$u = c$$
$$\therefore \tan\theta = 1, \quad \theta = 45^\circ$$
(World line of light wave)

- Now consider S' → frame axis moves with vel. v w.r.t S frame along $(x - x')$ axis

- at $t = 0$, both origin co-incide

$\therefore \quad x' = 0$

$\therefore$ from (1) $\boxed{x' = 0} \Rightarrow \boxed{x = \beta w}$

$\boxed{x = \frac{v}{c} \cdot ct = vt}$

→ time axis for S'-frame

Similarly

$w' = 0 \Rightarrow$ Space axis in S'

$w' = 0 \Rightarrow \boxed{w = \beta x}$

S	S'
x-axis $\Rightarrow w = 0$	x'-axis $\Rightarrow w' = 0$
w-axis $\Rightarrow x = 0$	w' - " $\Rightarrow x' = 0$

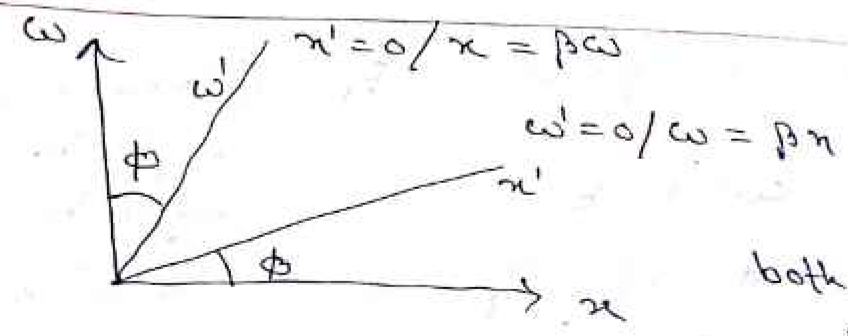


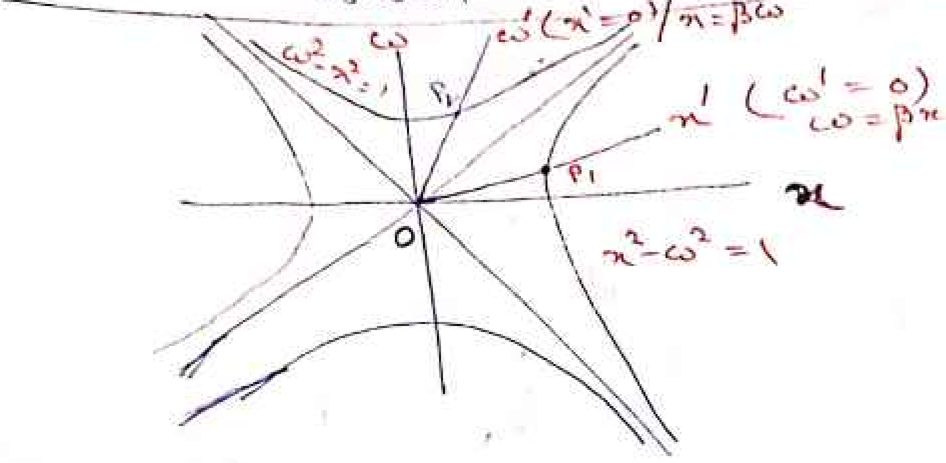
Fig 2

both slope ϕ are same. ($y = mx$, $m = \beta$)

(*) ⇒ From Fig (2)

L.T ⇒ ~~for~~ for transform from an orthogonal system (x, y, z, t) to a non-orthogonal system (x', y', z', t')

Let us consider the following diagram



we ~~consider~~ draw two branches of hyperbola $\omega^2 - x^2 = 1$

and two branches of hyperbola $x^2 - \omega^2 = 1$

in the diagram of $(x-\omega)$ and $(x'-\omega')$.

This lines will asymptotically approach 45^0 light wave world-lines,

consider a point P_1 (intersection of right branch of hyperbola $x^2-\omega^2=1$ and x'-axis.)

What will be Co-ord of P_1

$$x^2 - \omega^2 = 1 \quad , \quad \omega' = 0 \Rightarrow \omega = \beta x$$
$$x^2 - \beta^2 x^2 = 1$$
$$x^2 = \frac{1}{1-\beta^2} \Rightarrow \boxed{x = \frac{1}{\sqrt{1-\beta^2}}}$$

2 $\leftarrow$ $\boxed{\omega = \beta x = \frac{\beta}{\sqrt{1-\beta^2}}}$

From (1) $\Rightarrow$ $x' = \dfrac{x-\beta\omega}{\sqrt{1-\beta^2}}$, $\omega' = \dfrac{\omega-\beta x}{\sqrt{1-\beta^2}}$ (32)

(2) $\Rightarrow$ $x = \dfrac{1}{\sqrt{1-\beta^2}}$, $\omega = \dfrac{\beta}{\sqrt{1-\beta^2}}$

$$x' = \frac{x-\beta\omega}{\sqrt{1-\beta^2}} = \frac{\frac{1}{\sqrt{1-\beta^2}} - \beta.\frac{\beta}{\sqrt{1-\beta^2}}}{\sqrt{1-\beta^2}} = 1$$

$$\omega' = \frac{\omega-\beta x}{\sqrt{1-\beta^2}} = \frac{\beta}{\sqrt{1-\beta^2}} - \beta.\frac{1}{\sqrt{1-\beta^2}}$$
$$= 0.$$

$$\boxed{\begin{aligned} x' &= 1 \\ \omega' &= 0 \end{aligned}} \rightarrow \text{S}'\text{-frame}$$

$\therefore$ $OP_1 \longrightarrow$ gives unit length along x'-axis.

Similarly P_2 (on ω'-axis)

is the intersection of $\omega^2 - x^2 = 1$

and $x = \beta\omega$

(3) $\Leftarrow\Rightarrow$ $\omega = \dfrac{1}{\sqrt{1-\beta^2}}$, $x = \dfrac{\beta}{\sqrt{1-\beta^2}}$

From (3) and (1)

$$\boxed{\begin{aligned} \omega' &= 1 \\ x' &= 0 \end{aligned}}$$

$OP_2 \rightarrow$ unit time along ω'-axis.

- → At any other point x

$$w^2 - x^2 = 1$$

$$c^2 t^2 - x^2 = 1$$

$$c^2 \left(t^2 - \frac{x^2}{c^2}\right) = 1$$

$$c^2 \tau^2 = 1$$

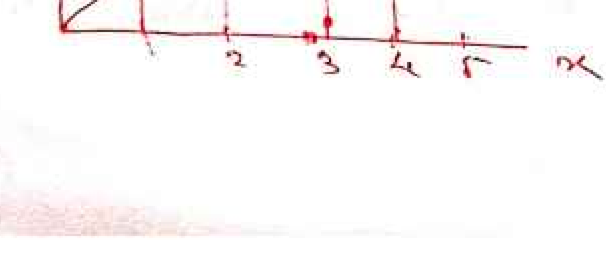
$$c\tau = 1$$

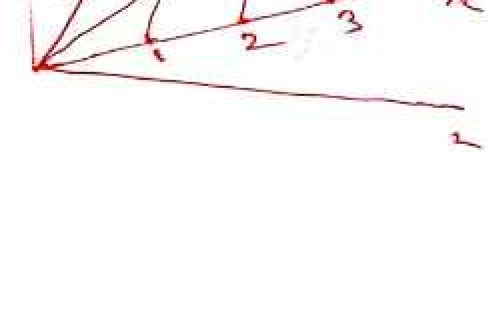
Thus points on the upper hyperbola give unit time on clock at rest in S'-frame.

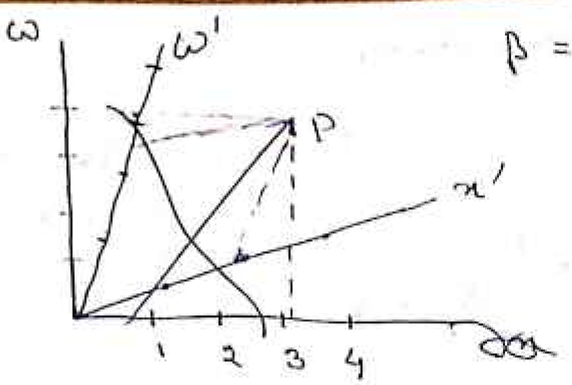
i.e. proper time in S'

- Whatever the relative velocity of S' ~~to~~ w.r.t S, intersects gives unit time.

- Similarly right hyperbola gives unit length.







$\beta = 0.5$ (say).

$x = 3, \ \omega = 2.5$

$x' = 2, \ \omega' = 1.5$

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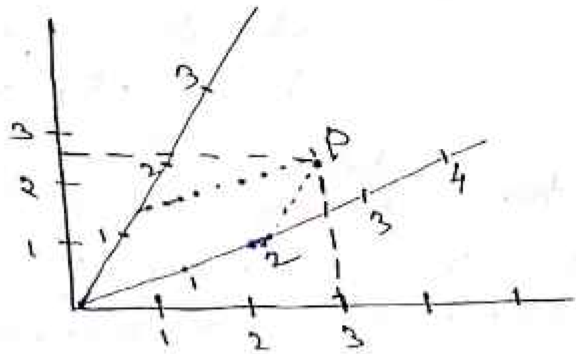
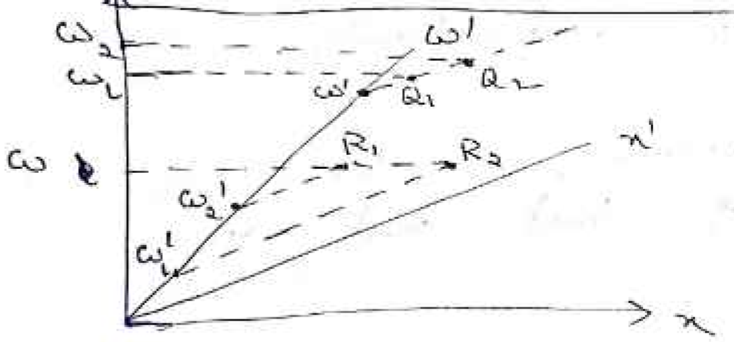
you can check it from equtⁿ (1).



graphical representⁿ of L.T

(Here unit length and unit time interval on the axis of these frames according to hyperbolic calibration curve.)

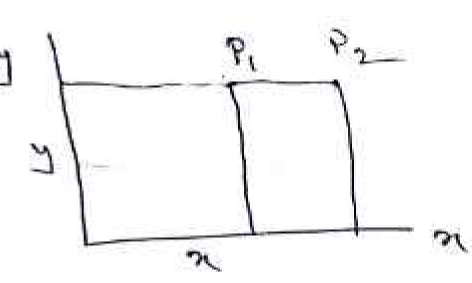
Simultaneity, Contraction, Dilation.



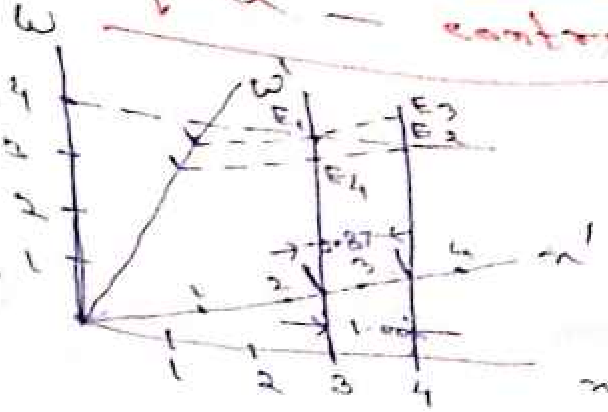
Simultaneity

$Q_1, Q_2 \rightarrow$ events are simultaneous in S' — frame $(t = \omega')$, but not in S — frame $(t_1 = \omega_1, t_2 = \omega_2)$

R_1, R_2 are simultaneous in S — frame $(t = \omega)$ — ~~total~~, but not in S' — frame $(t_1' = \omega_1', \ t_2' = \omega_2')$



Space — contraction.



Let a metre stick is in S-frame betn $x=3$, $x=4$

$\beta = 0.5$

in x'-axis it 0.27 m

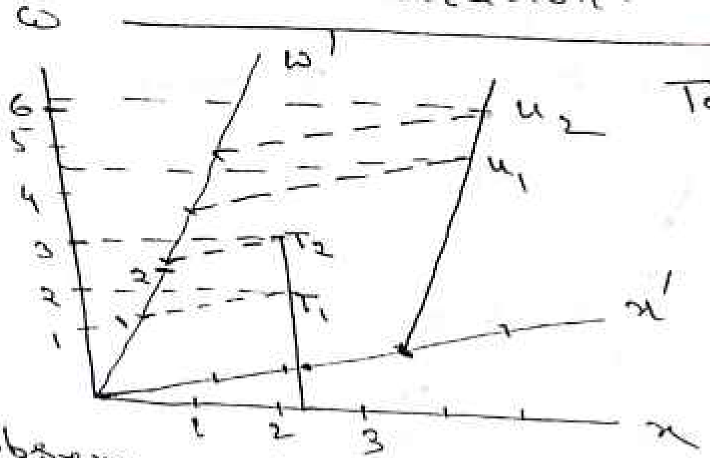
E1, E2 → two ~~events~~ points in S-frame at $x=3$, and $x=4$

two events E1, E2 are simultaneous in S-frame (A observer seen two event at same time $t=4$)

But an observer in S' frame will see these in E3 and E4 respectively.

So in S-frame, events are simultaneous but not in S'-frame.

Time Dilation.



Two events T_1, T_2
at $x = 2 \cdot 3$
$t_1 = 2$, $t_2 = 3$
(S- frame)
$\Delta t = 1$ unit

S- observer sees moving clock ~~observer~~ in S'-frame appears slowed down

S'— frame

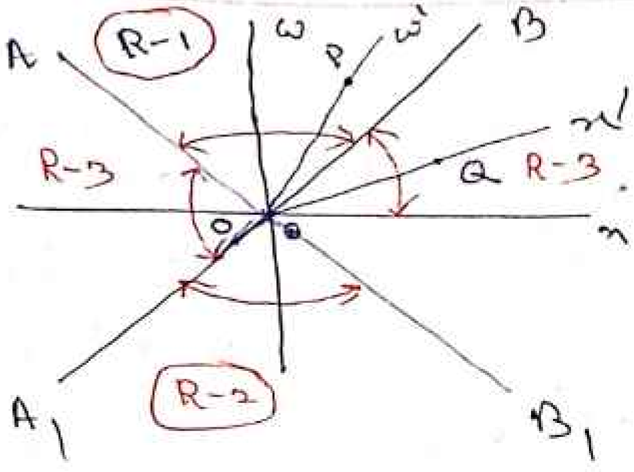
$\Delta t' > \Delta t$
$\Delta t'$ greater than unity.

u_1, u_2 events (at same point $x' = 3$)
in S'— frame, $\Delta t' = 1$
$\Delta t > 1$

Here S moves with v velocity w.r.t S' frame.

S'— observer sees $\Delta t > 1$ (S-frame)
i.e moving S- clock appears slowed down.

Light cone, causality, (Present, Past, Future)



consider regions AOB (R-1) and A_1OB_1 (R-2).
Through OP, ω'– axis are drawn and OQ, x' – axis are drawn.

→ events O (t=0) and P (t=t) occur at same place with a time interval.
So, we can say event P follows O.

→ So, upper half AOB is absolute future.

→ Similarly A_1OB_1 → absolute past w.r.t O (present, t = 0)

→ In this two region (upper cone AOB and lower cone A_1OB_1) has a definite time order of events w.r.t O.

→ Here we always find a frame for events at O and P which occur at the same place (38), So, a single clock can measure absolutely the time order of the events at same place.

→ Now consider
~~But from~~ BOB, and AOA, region, (R-3) (R-3')

→ O, Q are two ~~of~~ space-time pt.

O, Q → Same time (x'-axis)
→ or, simultaneous → present in one inertial frame.

→ But not simultaneous in other inertial frame.

→ There is no absolute time-order of these events (relative time-order)

→ This region is present,

→ In region (1) and (2)
there is no definite space ~~separation~~ order relative to O.
these two regions are said to be "time-like"

absolute future
absolute past

→ In region (3) → no definite time order (space separate of two events). This is 'space-like'.

$\rightarrow$

OQ $\rightarrow$ space like
OP $\rightarrow$ time "

consider quantity

$$c^2\tau^2 = c^2t^2 - (x^2+y^2+z^2)$$

or $$\tau^2 = t^2 - \frac{x^2+y^2+z^2}{c^2}$$

For simplicity we consider only x - co-ord

$$\tau^2 = t^2 - \frac{x^2}{c^2}$$

or, $$c^2\tau^2 = c^2t^2 - x^2$$

or, $$\boxed{c^2\tau^2 = \omega^2 - x^2} \rightarrow \text{invariant}$$

or, $$-c^2\tau^2 = x^2 - \omega^2$$

or, $$\boxed{\sigma^2 = x^2 - \omega^2} \rightarrow \text{invariant}$$

$$\boxed{\omega^2 - x^2 = \omega'^2 - x'^2 \quad / \quad x^2 - \omega^2 = x'^2 - \omega'^2}$$

in region R-1, R-2

$$\omega > x$$

$c^2\tau^2 \rightarrow$ +ve

$\tau \rightarrow$ real

$\hookrightarrow$ proper time is real

$\downarrow$

Time - like

in region R-3,

$x > c t$

$\sigma \rightarrow$ real

$\tau \rightarrow$ imaginary

$\hookrightarrow$ proper time imaginary (space-like).

World line of light cone.

$x = ct$

$c^2 \tau^2 = 0$, $\Rightarrow$ corresponds to $v = c$

$\Rightarrow$ in the R-3 $\rightarrow$ no definite time order.

event O precede event Q in one frame

event Q " event O in another frame.

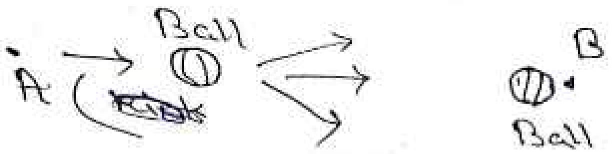
(cause $\Rightarrow$ $v > c$ Here)

So, a light pulse emitted by O-point reaches a point say Q later.)

i.e time-diff betn two events is less than the time needed for light pulse to cover the spatial dist). This is not possible according to Einstein's

Postulate of maximum velocity c.

No physical signal can travel faster ~~with~~ than c.



$v > c$

↳ vel. of ball

(Ball reaches before light rays?)

Conclusion in R-3

No ~~ϕ~~ space-time event in R-3 exists

Transformation of velocities

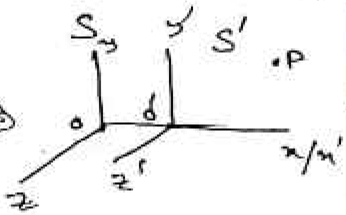
$S, S' \longrightarrow$ Frames of reference.

$S' \longrightarrow$ moves with v along $(x\text{-}x')$ axis

$P \longrightarrow$ a moving particle, whose velocity in S-frame is u, and in S'-frame u'.

$\Rightarrow$ our aim is to find out the relation between u and u'.

motion of P in S $\Rightarrow$



$$x = x(t)$$
$$y = y(t)$$
$$z = z(t)$$

in S',

$$x' = x'(t')$$
$$y' = y'(t')$$
$$z' = z'(t')$$

in $S \Rightarrow$ comp. of velocity of the particle in x, y, z directi

$$u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt}.$$

$$u = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

Similarly in S' frame

$$u_x' = \frac{dx'}{dt'}, \quad u_y' = \frac{dy'}{dt'}, \quad u_z' = \frac{dz'}{dt'}$$

$$u' = \sqrt{u_x'^2 + u_y'^2 + u_z'^2}$$

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right).$$
$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$u_x' = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma\left(dt - v\frac{dx}{c^2}\right)}$$

$$= \frac{\frac{dx}{dt} - v}{1 - \frac{v}{c^2}u_x} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)}.$$

$$= \frac{dy/dt}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)}$$

$$= \frac{u_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

$$u_z' = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}.$$

$$u' = \sqrt{u_x'^2 + u_y'^2 + u_z'^2}.$$

$$= \frac{1}{1 - \frac{u_x v}{c^2}}\left\{(u_x - v)^2 + \frac{u_y^2}{\gamma^2} + \frac{u_z^2}{\gamma^2}\right\}^{\frac{1}{2}}.$$

$$= \frac{1}{1 - \frac{u_x v}{c^2}}\left[(u_x - v)^2 + (u_y^2 + u_z^2)\left(1 - \frac{v^2}{c^2}\right)\right]^{\frac{1}{2}}$$

$$= \frac{1}{1-\frac{u_x v}{c^2}}\left[u_x^2 - 2u_x v + v^2 + u_y^2 + u_z^2 - \frac{v^2}{c^2}\left(u_y^2+u_z^2\right)\right]^{\frac{1}{2}}$$

$$= \frac{1}{1-\frac{u_x v}{c^2}}\left[u^2 - 2u_x v + v^2 - \frac{v^2}{c^2}\left(u^2 - u_x^2\right)\right]^{\frac{1}{2}}$$

$$u' = \frac{1}{1-\frac{u_x v}{c^2}}\left[u^2 - 2u_x v + v^2 - \frac{u^2 v^2}{c^2} + \frac{v^2 u_x^2}{c^2}\right]^{\frac{1}{2}}$$

Now, if particle is moving along x-axis

$u_x = u, \quad u_y = u_z = 0, \quad u_y' = u_z' = 0.$

$$u' = \frac{1}{1-\frac{uv}{c^2}}\left[u^2 - 2uv + v^2 - \frac{u^2v^2}{c^2} + \frac{u^2v^2}{c^2}\right]^{\frac{1}{2}}$$

$$= \frac{u-v}{1-\frac{uv}{c^2}}$$

conversely $\quad u = \dfrac{u'+v}{1+\frac{u'v}{c^2}}$

$v \ll c, \quad \frac{v}{c^2} \ll 1$

$\therefore \; u' = (u - v)$ (classical result).

if $\quad u' = c$

$$u = \frac{c+v}{1+\frac{cv}{c^2}} = c.$$

Variation of mass with velocity

- Mass → quantity contained in a body.
  → more correctly, as the resistance it offers to an external force which tends to change its dynamical state. (inertial mass of the given body).
→ may be introduced through the gravitational attraction of two bodies. (gravitational mass).

→ In Newtonian Mechanics, the mass of a body is taken as absolute (not dependent on its state of motion w.r.t the observer.)

According to 2nd law of motion equal forces impressed upon a body would produce in it equal acceleration whatever be the velocity of the body. So, velocity of the body would go on increasing indefinitely with a cont. rate for the continuous application of a force on it. But there is a natural upper limit of the velocity of body i.e. c → vel. of light.

So, we can conclude that the inertia of a body must increase with velocity, tends to infinity as the velocity of the body approaches (c).

→ In this way, we can say, the force acting on the body will produce constant acceleration in initial stage but its influence becomes ~~greater~~ gradually less and less as velocity becomes more and more and ultimately its effect vanishes as the velocity tends to the vel. of light in vacuum.

So, inertial mass of a body cannot be taken as a constant quantity but it increases with its velocity tending to infinity as its velocity approaches to c.

$$m_u = m_0 f(u)$$

m_0 → mass of the body at rest

m_u → " " " " in motion with velocity u.

$f(u)$ → depends on u

$f(u) \to 1$, as $u \to 0$

$\to \infty$, " $u \to c$

Momentum of the body is.

$$P = m_u u$$

conservation of momentum

$\Rightarrow$ vector sum of momenta of mutually interacting bodies remains unchanged.

Principle of relativity $\rightarrow$
Laws are valid in all inertial frames.

Using this fact we can calculate $f(u)$

- $\rightarrow$ Let us consider a system of n-particles.

Velocity of ith particle in S, S' frames are u_i, u_i'

$$u_i = \frac{1}{1 + \frac{u'_{xi} v}{c^2}} \left[u_i'^2 + 2 u'_{xi} v + v^2 - \frac{v^2}{c^2} \left(u_i'^2 - u_{xi}'^2 \right) \right]^{\frac{1}{2}}$$

$$\frac{u_i^2}{c^2} = \frac{1}{\left(1+\frac{u'_{x_i}v}{c^2}\right)^2}\left[\frac{u_i'^2}{c^2} + \frac{2u'_{x_i}v}{c^2} + \frac{v^2}{c^2} - \frac{v^2}{c^4}\left(u_i'^2 - u_{x_i}'^2\right)\right]$$

$$= \frac{1}{\left(1+\frac{u'_{x_i}v}{c^2}\right)^2}\left[\frac{u_i'^2}{c^2}\left(1-\frac{v^2}{c^2}\right) + \frac{2u'_{x_i}v}{c^2} + \frac{v^2}{c^2} + \frac{v^2u_{x_i}'^2}{c^4}\right]$$

$$1 - \frac{u_i^2}{c^2} = \frac{\left(1+\frac{u'_{x_i}v}{c^2}\right)^2 - \left[\frac{u_i'^2}{c^2}\left(1-\frac{v^2}{c^2}\right) + \frac{2u'_{x_i}v}{c^2} + \frac{v^2}{c^2} + \frac{v^2u_{x_i}'^2}{c^4}\right]}{\left(1+\frac{u'_{x_i}v}{c^2}\right)^2}$$

$$= \frac{1 + \cancel{\frac{2u'_{x_i}v}{c^2}} + \cancel{\frac{u_{x_i}'^2v^2}{c^4}} - \frac{u_i'^2}{c^2}\left(1-\frac{v^2}{c^2}\right) - \cancel{\frac{2u'_{x_i}v}{c^2}} - \frac{v^2}{c^2} - \cancel{\frac{v^2u_{x_i}'^2}{c^4}}}{(\quad)^2}$$

$$1 - \frac{u_i^2}{c^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u_i'^2}{c^2}\right)}{\left(1+\frac{u'_{x_i}v}{c^2}\right)^2}$$

$$\frac{1}{\sqrt{1-\frac{u_i^2}{c^2}}} = \frac{1+\frac{u'_{xi}v}{c^2}}{\sqrt{1-\frac{u_i'^2}{c^2}}\sqrt{1-\frac{v^2}{c^2}}}$$

$$\gamma_i = \gamma'_i \gamma \left(1+\frac{u'_{xi}v}{c^2}\right). \rightarrow (A)$$

we consider S' moves with v along x — axis,

$$\therefore u'_{xi} v = \vec{u}'_i \cdot \vec{v}$$

$$\boxed{\gamma_i = \gamma'_i \gamma \left(1+\frac{u'_i \cdot v}{c^2}\right)} \rightarrow (A1)$$

$$u_i = u_{xi}\,i + u_{yi}\,j + u_{zi}\,k.$$

$$= \frac{u'_{xi}+v}{1+\frac{u'_{xi}v}{c^2}}\hat{i} + \frac{u'_{yi}}{\gamma\left(1+\frac{u'_{xi}v}{c^2}\right)}\hat{j} + \frac{u'_{zi}}{\gamma\left(\quad\right)}\hat{k} \rightarrow (B)$$

multiply (A) and (B)

$$\gamma_i u_i = \gamma'_i \gamma \left(1+\frac{u'_{xi}v}{c^2}\right) \cdot \left[\frac{u'_{xi}+v}{1+\frac{u'_{xi}v}{c^2}}\hat{i} + \frac{u'_{yi}}{\gamma\left(1+\frac{u'_{xi}v}{c^2}\right)}\hat{j} + \frac{u'_{zi}\hat{k}}{\gamma\left(1+\frac{u'_{xi}v}{c^2}\right)}\right]$$

$$= \gamma_i' \left[\gamma (u'_{xi} + v)\hat{i} + u'_{yi}\hat{j} + u'_{zi}\hat{k} \right]$$

$$= \gamma_i' \left[\vec{u}_i' + (\gamma - 1)\, u'_{xi}\hat{i} + \gamma v \hat{i} \right]$$

$$\gamma_i \vec{u}_i = \gamma_i' \left[\vec{u}_i' + (\gamma-1)\frac{(\vec{u}_i' \cdot \vec{v})\vec{v}}{v^2} + \gamma \vec{v} \right]$$

$$M = \sum_i m_i$$

$$\vec{P} = \sum_i (m_i u_i)$$

$$\vec{P} = \sum_i m_i \frac{\gamma_i'}{\gamma_i} \left\{ \vec{u}_i' + (\gamma-1)\frac{(\vec{u}_i' \cdot \vec{v})\vec{v}}{v^2} + \gamma \vec{v} \right\}$$

$$= \sum_i m_i' \vec{u}_i' + (\gamma - 1)\vec{v}\, \frac{\sum_i (m_i' u_i') \cdot \vec{v}}{v^2} + \gamma \vec{v} \sum_i m_i' \longrightarrow (e)$$

$$\boxed{m_i' = m_i \frac{\gamma_i'}{\gamma_i}}$$

$$\boxed{\frac{m_i'}{\gamma_i'} = \frac{m_i}{\gamma_i}} = m_{i0}$$

$m_{i0} \rightarrow$ independent of anything related to S and S'

$$M = \sum_i m_i = \sum m_i' \frac{\nu_i}{\nu_i'}$$

[from equ (A1)]

$$= \sum_i m_i' \, \nu \left(1 + \frac{u_i' \cdot v}{c^2}\right)$$

$$= \nu \left[\sum_i m_i' + \left\{ \sum_i \frac{(m_i' \vec{u}_i') \cdot \vec{v}}{c^2} \right\} \right]$$

Similar transformation as. general L.T

$$M' = \sum m_i'$$

$$p' = \sum_i m_i' u_i'$$

$$m_i' = m_i \frac{\nu_i'}{\nu_i}$$

$$\frac{m_i}{\nu_i} = \frac{m_i'}{\nu_i'} \rightarrow \text{indep of } (u_i', v)$$

$$= m_{i0}$$

$\hookrightarrow$ indep of (u_i, v)

$$m_i = m_{i0} \, \nu_i$$

$$m_i' = m_{i0} \, \nu_i'$$

$$m = m_0 \nu = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$